UK SENIOR MATHEMATICAL CHALLENGE

November 6th 2012

EXTENDED SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations. (The short solutions have been appended and appear at the end of this document.)

The Senior Mathematical Challenge (SMC) is a multiple choice contest, in which you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to: enquiry@ukmt.co.uk

or by post to: SMC Solutions, UKMT Maths Challenges Office, School of Mathematics, University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	B	D	B	C	C	D	C	C	E	D	E	B	D	A	A	B	A	E	E	D	B	C	B	B

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1.	Which of the following cannot be written as the sum of two prime numbers?	
	A 5 B 7 C 9 D 10 E 11	

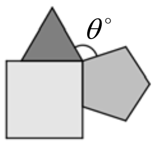
Solution: **E**

If the odd number 11 is the sum of two prime numbers, it must be the sum of the only even prime, 2, and an odd prime. But $11 = 2 + 9$ and 9 is not prime. So 11 is not the sum of two prime numbers. On the other hand, $5 = 2 + 3$, $7 = 2 + 5$, $9 = 2 + 7$ and $10 = 3 + 7$. Therefore, these are all the sum of two prime numbers.

Extension Problem

1.1 In 1742 the German Mathematician Christian Goldbach (1690-1764) made the conjecture that every even number greater than 2 is the sum of two primes. The *Goldbach Conjecture*, as it is now called, has, so far, not been proved, but no counterexample to it has been found. Investigate this conjecture by finding the even number in the range from 4 to 100 which can be expressed as the sum of two prime numbers in the largest number of ways.

2. The diagram shows an equilateral triangle, a square and a regular pentagon which all share a common vertex. What is the value of θ?



A 98   B 102   C 106   D 110   E 112

Solution: **B**

The interior angles of an equilateral triangle, a square and a regular pentagon are 60°, 90° and 108°, respectively. Hence $\theta = 360 - (60 + 90 + 108) = 360 - 258 = 102$.

3. The price of my favourite soft drink has gone up in leaps and bounds over the past ten years. In four of those years it has leapt up by 5p each year, whilst in the other six years it has bounded up by 2p each year. The drink cost 70p in 2002. How much does it cost now?

A £0.77   B £0.90   C £0.92   D £1.02   E £1.05

Solution: **D**

The cost has risen by 5p four times, and by 2p six times. So the total price rise in pence has been $(4 \times 5 + 6 \times 2) = 32$. Therefore the price now is $(70 + 32)\text{p} = 102\text{p} = £1.02$.

4. According to one astronomer, there are one hundred thousand million galaxies in the universe, each containing one hundred thousand million stars. How many stars is that altogether?

A 10^{13} B 10^{22} C 10^{100} D 10^{120} E 10^{121}

Solution: **B**

One hundred thousand is 10^5 and one million is 10^6. So one hundred thousand million is $10^5 \times 10^6 = 10^{5+6} = 10^{11}$. Therefore the total number of stars is $10^{11} \times 10^{11} = 10^{11+11} = 10^{22}$.

5. All six digits of three 2-digit numbers are different. What is the largest possible sum of three such numbers?

A 237 B 246 C 255 D 264 E 273

Solution: **C**

Suppose that the six digits that are used are *a, b, c, d, e* and *f* and that we use them to make the three 2-digit numbers '*ad*', '*be*' and '*cf*'. The sum of these numbers is $(10a+d)+(10b+e)+(10c+f)$, that is $10(a+b+c)+(d+e+f)$. To make this as large as possible, we need to choose *a*, *b* and *c so* that $a+b+c$ is as large as possible, and then *d*, *e* and *f* from the remaining digits so that $d+e+f$ is large as possible. Clearly, this is achieved by choosing *a, b, c* to be 7, 8, 9, in some order, and then *d, e, f* to be 4, 5, 6 in some order. So the largest possible sum is $10(7+8+9)+(4+5+6)=255$. [For example, the three 2-digit numbers could be 74, 85 and 96.]

6. What is the sum of the digits of the largest 4-digit palindromic number which is divisible by 15? [Palindromic numbers read the same backwards and forwards, e.g. 7227.]

A 18 B 20 C 24 D 30 E 36

Solution: **C**

A 4-digit palindromic number has the form deed where d and e are digits. This number is divisible by 15 if and only if it is divisible by both 3 and 5. It will be divisible by 5 if and if only if d is 0 or 5, but the first digit of a number cannot be 0, so d is 5, and the number has the form 5ee5.

A number is divisible by 3 if and only if the sum of its digits is divisible by 3. So 5ee5 is divisible by 3 if and only if $10+2e$ is divisible by 3, that is, when e is 7, 4 or 1. So the largest 4-digit palindromic number which is divisible by 15 is 5775. The sum of the digits of 5775 is $5+7+7+5=24$.

Extension problems

6.1 Find the largest 4-digit palindromic number which is a multiple of 45.

6.2 Find the largest 5-digit and 6-digit palindromic numbers which are multiples of 15.

6.3 The argument above uses the fact that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3. Explain why this test works.

6.4 Find a similar test for divisibility by 9.

6.5 Find a test, in terms of its digits, to decide when a positive integer is divisible by 11.

7. Given that $x+y+z=1$, $x+y-z=2$ and $x-y-z=3$, what is the value of *xyz*?

A -2 B $-\frac{1}{2}$ C 0 D $\frac{1}{2}$ E 2

Solution: **D**

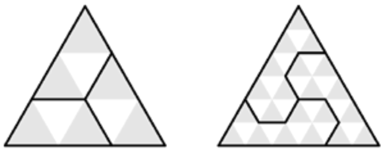
Adding the equations, $x+y+z=1$ and $x-y-z=3$, gives $2x=4$, from which it follows that $x=2$. Adding the equations $x+y+z=1$ and $x+y-z=2$, gives $2(x+y)=3$. Hence $x+y=\frac{3}{2}$. Therefore, as $x=2$, it follows that $y=-\frac{1}{2}$. Since $x+y+z=1$, it now follows that $z=1-(x+y)=1-\frac{3}{2}=-\frac{1}{2}$. Therefore, $xyz=2\times-\frac{1}{2}\times-\frac{1}{2}=\frac{1}{2}$.

8. The diagrams below show four types of tile, each of which is made up of one or more equilateral triangles. For how many of these types of tile can we place three identical copies of the tile together, without gaps or overlaps, to make an equilateral triangle?

A 0 B 1 C 2 D 3 E 4

Solution: **C**

An equilateral triangle may be divided into 1, 4, 9, 16 ,… , that is, a square number, of equilateral triangles of the same size. Three tiles, each made up of k smaller equilateral triangles of the same size, contain altogether 3*k* of the smaller triangles. So, if it is possible to place three copies of the tile to make an equilateral triangle, then 3*k* must be a square number. For the above tiles *k* takes the values 1, 2, 3 and 12, respectively. Of these, only for $k = 3$ and $k = 12$ do we have that 3*k* is a square. So the only possible cases where three of the above tiles can be rearranged to make an equilateral triangle are the two tiles on the right. We see from the diagrams above that in both cases we can use three of the tiles to make an equilateral triangle. So there are 2 such cases.



Extension problem

8.1 In the argument above we have stated, *but not proved*, that if an equilateral triangle is divided into smaller equilateral triangles of the same size, then the number of the smaller triangles is always a perfect square. Prove that this is correct.

8.2 Determine a necessary and sufficient condition for a positive integer, *k*, to be such that 3*k* is a square. [Ask your teacher if you are not sure what is meant by a *necessary and sufficient condition*.]

8.3 Is it possible to find a tile which is made up of *k* equilateral triangles of the same size, where 3*k* is a square, but three copies of the tile cannot be placed together, without gaps or overlaps, to make an equilateral triangle?

9. Pierre said, “Just one of us is telling the truth”. Qadr said, “What Pierre says is not true”. Ratna said, “What Qadr says is not true”. Sven said, “What Ratna says is not true”. Tanya said, “What Sven says is not true”.

How many of them were telling the truth?

A 0 B 1 C 2 D 3 E 4

Solution: **C**

If Pierre is telling the truth, everyone else is not telling the truth. But, also in this case, what Qadr said is not true, and hence Ratna is telling the truth. So we have a contradiction.

So we deduce that Pierre is not telling the truth. So Qadr is telling the truth. Hence Ratna is not telling the truth. So Sven is also telling the truth, and hence Tanya is not telling the truth. So Qadr and Sven are telling the truth and the other three are not telling the truth.

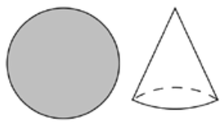
10.	Let N be the smallest positive integer whose digits add up to 2012. What is the first digit of $N+1$? A 2 B 3 C 4 D 5 E 6

Solution: **E**

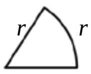
If *N* is the smallest positive integer whose digits add up to 2012, it will have the smallest possible number of digits among numbers whose digits add up to 2012. This means that, as far as possible, each digit should be a 9. So *N* will have the form $d\underbrace{999...999}_{k}$, where *k* is a positive integer and *d* is a non-zero digit. Now, $2012 = 223 \times 9 + 5$. So $k = 223$ and $d = 5$. Hence $N = 5\underbrace{999...999}_{223}$.

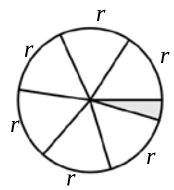
Therefore $N + 1 = 6\underbrace{000...000}_{223}$. So the first digit of $N+1$ is a 6.

11.	Coco is making clown hats from a circular piece of cardboard. The circumference of the base of each hat equals its slant height, which in turn is equal to the radius of the piece of cardboard. What is the maximum number of hats that Coco can make from the piece of cardboard? A 3 B 4 C 5 D 6 E 7



Solution: **D**

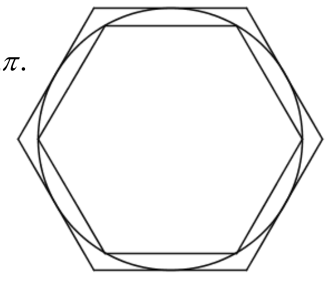




Let the radius of the circle be *r*. This must also be the slant height of each hat. If we flatten out a hat we get a sector of a circle in which the circular part of the boundary has length *r*, as shown in the diagram on the left. The circumference of a circle of radius *r* has length $2\pi r$. Since $3 < \pi < 3.5$, it follows that $6r < 2\pi r < 7r$. So Coco can cut out 6 sectors from the circular piece of card as shown, each of which can be made into a hat. The area of card needed for each hat is $\frac{1}{2}r^2$ and the area of the circle is πr^2. As $\pi r^2 < 7(\frac{1}{2}r^2)$, Coco cannot make 7 hats. So the maximum number of hats that Coco can make is 6.

Extension Problems

In this solution we have used the fact that $3 < \pi < 3.5$. How do we know that this is true? Here we ask you to show this, starting with the definition of π as the ratio of the circumference of a circle to its diameter (and making one geometrical assumption).



11.1 Suppose we have circle with diameter 1. Its radius is therefore 2π. We consider a regular hexagon with its vertices on the circle, and a regular hexagon which touches the circle, as shown in the diagram. Use the assumption that the circumference of the circle lies between the perimeters of the two hexagons to show that $3 < \pi < 2\sqrt{3}$. Deduce that $3 < \pi < 3.5$.

11.2 The method of approximating the circumference of a circle by a regular polygon was discovered independently in more than one culture. The Greek mathematician Archimedes who lived from 287BC to 212BC used regular polygons with 96 sides to obtain the approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The Chinese mathematician Liu Hui, who lived around 260CE, obtained the approximation $3\frac{4407}{62500} < \pi < 3\frac{8919}{62500}$. Use the approximate value $\pi \approx 3.14159$ to estimate the percentage errors in these approximations.

[Today, using iterative methods that converge very rapidly, and powerful computers, π has been calculated to billions of decimal places.]

12. The number 3 can be expressed as the sum of one or more positive integers in four different ways:

$$3; \quad 1+2; \quad 2+1; \quad 1+1+1.$$

In how many ways can the number 5 be so expressed?

A 8 B 10 C 12 D 14 E 16

Solution: **E**

By systematically listing all the possibilities, we see that the number 5 may be written as a sum of one or more positive integers in the following 16 ways:

$$5;\ 1+4;\ 4+1;\ 2+3;\ 3+2;\ 3+1+1;\ 1+3+1;\ 1+1+3;\ 1+2+2;\ 2+1+2;$$
$$2+2+1;\ 1+1+1+2;\ 1+1+2+1;\ 1+2+1+1;\ 2+1+1+1;\ 1+1+1+1+1.$$

Extension Problems

12.1 In how many ways can the number 4 be expressed as the sum of one or more positive integers?

12.2 To generalize these results we need a better method than listing all the possibilities. From the cases $n=3,4$ and 5, it seems reason to guess that a positive integer *n* can be expressed as the sum of one or more positive integers in 2^{n-1} ways. Can you prove that this conjecture is correct?

12.3 We can also consider the number of different ways of expressing a positive integer, *n*, as the sum of positive integers *when the order does not matter*. For example we count $1+1+2$ as being the same as $1+2+1$. These arrangements, where the order does not matter, are called *partitions*. You can see that there are just 7 different partitions of 5, namely, 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$ and $1+1+1+1+1$. Count the number of different partitions of *n*, for $1 \le n \le 8$. [There is no simple formula for the number of different partitions of *n*.]

13. A cube is placed with one face on square 1 in the maze shown, so that it completely covers the square with no overlap. The upper face of the cube is covered in wet paint. The cube is 'rolled' around the maze, rotating about an edge each time, until it reaches square 25. It leaves paint on all of the squares on which the painted face lands, but on no others. The cube is removed on reaching the square 25. What is the sum of the numbers on the squares which are now marked with paint?

5	6	7	8	9
4	19	20	21	10
3	18	25	22	11
2	17	24	23	12
1	16	15	14	13

A 78 B 80 C 82 D 169 E 625

Solution: **B**

T	S	**B**	L	T
F	S	**B**	L	F
B	S	F	L	**B**
S	S	**B**	L	S
T	S	**B**	L	T

We track the position of the side of the cube which is covered in wet paint. We imagine that the maze is horizontal, and that we are looking at it from the side with the squares which are marked with the numbers 1, 16, 15, 14 and 13. We use T, B, F, S, L and R for the Top, Bottom, Front, Stern (back), Left and Right sides of the cube, as we look at it, respectively.

Initially, the wet paint is on Top. We see from the diagram that the wet paint is on the bottom of the cube when the cube is on the squares labelled 3, 7, 11, 15, 20 and 24. The sum of these numbers is $3+7+11+15+20+24=80$.

14. Six students who share a house all speak exactly two languages. Helga speaks only English and German; Ina speaks only German and Spanish; Jean-Pierre speaks only French and Spanish; Karim speaks only German and French; Lionel speaks only French and English whilst Mary speaks only Spanish and English. If two of the students are chosen at random, what is the probability that they speak a common language?

A $\frac{1}{2}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{4}{5}$ E $\frac{5}{6}$

Solution: **D**

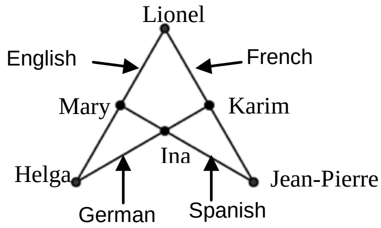
We can select two of the six students in 15 ways, as shown in the table. We have put a tick (✓) in each box corresponding to a pair of students who speak a common language, and a cross (✗) in each box corresponding to a pair of students who do not speak a common language.

	Helga E & G	Ina G & S	J-P F & S	Karim G & F	Lionel F & E
Ina G & S	✓				
J-P F & S	✗	✓			
Karim G & F	✓	✓	✓		
Lionel F & E	✓	✗	✓	✓	
Mary S & E	✓	✓	✓	✗	✓

We see that there are 12 ticks in the 15 boxes. Therefore, if two students are chosen at random, the probability that they speak a common language is $\frac{12}{15} = \frac{4}{5}$..

Alternative method
Note that each of the students has a common language with 4 out of 5 of the other students. It therefore follows that if two students are chosen at random, the probability that they speak a common language is $\frac{4}{5}$.

Note: We may represent the four languages by lines and the students by points which are on the lines corresponding to the languages they speak, as shown. This makes it easy to see that for each student, there is just one other student with whom they have no common language.



15. Professor Rosseforp runs to work every day. On Thursday he ran 10% faster than his usual average speed. As a result, his journey time was reduced by x minutes. How many minutes did the journey take on Wednesday?

A $11x$ B $10x$ C $9x$ D $8x$ E $5x$

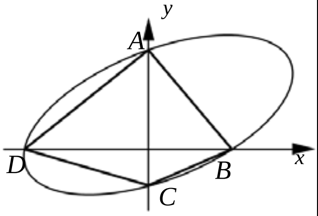
Solution: **A**

Suppose that the distance that Professor Rosseforp runs is d and his usual average speed (in terms of the units used for distance and minutes) is s. So, on Wednesday his journey takes him $\frac{d}{s}$ minutes. On Thursday his average speed was $\frac{110}{100}s = \frac{11}{10}s$ and so the journey takes him $\frac{d}{\frac{11}{10}s} = \frac{10d}{11s}$ minutes. We therefore have $\frac{d}{s} - \frac{10d}{11s} = x$. This gives $\frac{d}{11s} = x$ and hence $\frac{d}{s} = 11x$. So on Wednesday his journey took him $11x$ minutes.

Extension Problem

15.1 Now consider the general case where Professor Rosseforp runs the same distance p% slower on Monday than on Tuesday, and q% faster on Wednesday than on Tuesday. Suppose that his run on Wednesday took him x minutes less than his run on Monday. How many minutes did his run take on Tuesday?

16. The diagram shows the ellipse whose equation is $x^2 + y^2 - xy + x - 4y = 12$. The curve cuts the y-axis at points A and C and cuts the x-axis at points B and D.



What is the area of the inscribed quadrilateral *ABCD?*

A 28 B 36 C 42 D 48 E 56

Solution: **A**

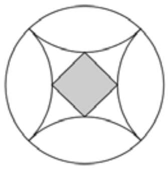
The graph cuts the *y*-axis at points where $x = 0$, and hence where $y^2 - 4y = 12$, that is, where $y^2 - 4y - 12 = 0$, that is, $(y+2)(y-6) = 0$, giving $y = -2$ and $y = 6$. Hence *A* is the point (0,6), *C* is the point $(0,-2)$ and so *AC* has length 8. Similarly, *B* and *D* are points where $y = 0$ and hence $x^2 + x - 12 = 0$, that is, $(x+4)(x-3) = 0$, giving $x = -4$ and $x = 3$. So *B* is the point $(3,0)$, *D* is the point $(-4,0)$, and so *BD* has length 7. The area of the inscribed quadrilateral *ABCD* is therefore $\frac{1}{2}AC.BD = \frac{1}{2}(8 \times 7) = 28$.

Extension Problems

16.1 The solution above takes it for granted that the area of the quadrilateral *ABCD* is half the product of the lengths of its diagonals. Give an example to show that this is *not true* for all quadrilaterals.

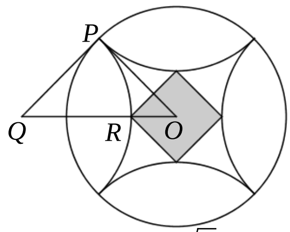
16.2 What is the special property of *ABCD* that makes it correct that in this case the area is half the product of the length of the diagonals? Give a proof to show that your answer is correct.

17. The diagram shows a pattern found on a floor tile in the cathedral in Spoleto, Umbria. A circle of radius 1 surrounds four quarter circles, also of radius 1, which enclose a square. The pattern has four axes of symmetry. What is the side length of the square?



A $\frac{1}{\sqrt{2}}$ B $2 - \sqrt{2}$ C $\frac{1}{\sqrt{3}}$ D $\frac{1}{2}$ E $\sqrt{2} - 1$

Solution: **B**



Let *O* be the centre of the circle, let *P* be one of the points where two of the quarter circles meet, let *Q* be the centre of one of these quarter circles, and let *R* be the vertex of the square that lies on *OQ*, as shown.

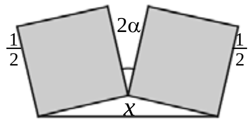
Then in the triangle *OPQ* there is a right angle at *P* and $OP = QP = 1$, and therefore, by Pythagoras' Theorem, $OQ = \sqrt{2}$. Since $QR = 1$, it follows that $OR = \sqrt{2} - 1$. It follows that the diagonal of the square has length $2(\sqrt{2} - 1)$. Using, Pythagoras' Theorem again, it follows that the side length of the square is $\frac{1}{\sqrt{2}}\left(2(\sqrt{2} - 1)\right) = 2 - \sqrt{2}$.

Extension Problem

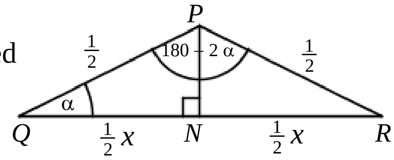
17.1 In the proof above we claim, by Pythagoras' Theorem, that if the diagonal of a square has length *x*, then its side length is $\frac{1}{\sqrt{2}}x$. Show that this is indeed a consequence of Pythagoras' Theorem.

18. The diagram shows two squares, with sides of length $\frac{1}{2}$, inclined at an angle 2α to one another. What is the value of *x*?

A $\cos\alpha$ B $\dfrac{1}{\cos\alpha}$ C $\sin\alpha$ D $\dfrac{1}{\sin\alpha}$ E $\tan\alpha$

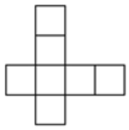


Solution: **A**



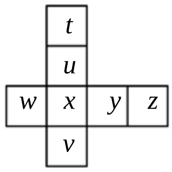
Consider the triangle at the bottom of the diagram. We have labelled the points *P*, *Q* and *R*, as shown. We let *N* be the point where the perpendicular from *P* to *QR* meets *QR*. .
Now, $\angle QPR = 360 - 90 - 2\alpha - 90 = 180 - 2\alpha$ **.** In the triangle *PQR* we have $PQ = PR = \frac{1}{2}$ as they are sides of the squares. It follows that the triangle *PQR* is isosceles and hence $\angle PQN = \angle PRN = \alpha$. So the triangles *PQN* and *PRN* are congruent. Hence $QN = NR = \frac{1}{2}x$. Therefore, from the triangle *PQN* we have $\cos\alpha = \dfrac{QN}{QP} = \dfrac{\frac{1}{2}x}{\frac{1}{2}} = x$ .That is, $x = \cos\alpha$.

19. The numbers 2, 3, 4, 5, 6, 7, 8 are to be placed, one per square, in the diagram shown so that the sum of the four numbers in the horizontal row equals 21 and the sum of the four numbers in the vertical column also equals 21. In how many different ways can this be done?



A 0 B 2 C 36 D 48 E 72

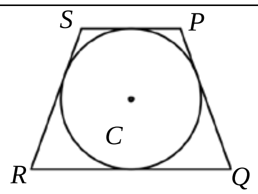
Solution: **E**



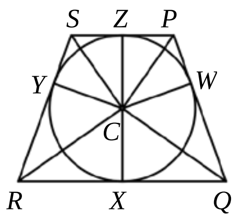
Let the numbers in the squares be *t*, *u*, *v*, *w*, *x*, *y* and *z*, as shown. We have $(w + x + y + z) + (t + u + x + v) = 21 + 21 = 42$, that is $(t + u + v + w + x + y + z) + x = 42$. But $t + u + v + w + x + y + z =$ $2 + 3 + 4 + 5 + 6 + 7 + 8 = 35$ and hence $x = 7$. So *t*, *u*, *v* are three numbers chosen from 2, 3, 4, 5, 6, 8 which add up to 14. It can be checked that there are just two choices for these numbers 2, 4, 8 and 3, 5, 6. If the numbers in the horizontal row are 2, 4, 8 and 7, there are 3 choices for *w* and then 2 choices for *y* and then 1 choice for *z*, making $3 \times 2 \times 1$ choices altogether, and then similarly 6 choices for *t*, *u*, *v* , making $6 \times 6 = 36$ combinations. Likewise there are 36 combinations where the numbers in the horizontal row are 3, 5, 6 and 7. This makes $36 + 36 = 72$ different ways altogether.

20. In trapezium *PQRS*, $SR = PQ = 25$ cm and *SP* is parallel to *RQ*. All four sides of *PQRS* are tangent to a circle with centre *C*. The area of the trapezium is 600 cm^2. What is the radius of the circle?

A 7.5cm B 8cm C 9cm D 10cm E 12cm



Solution: **E**



Let the radius of circle be *r* cm, and let the points where the trapezium touches the circle be *W*, *X*, *Y* and *Z* as shown. As the radii *CW*, *CX*, *CY* and *CZ* are perpendicular to the edges of the trapezium *PQ*, *QR*, *RS* and *SP*, respectively, the area of the trapezium, which is the sum of the areas of the triangles *CPQ*, *CQR*, *CRS* and *CSP*, is

$$\tfrac{1}{2}rPQ + \tfrac{1}{2}rQR + \tfrac{1}{2}rRS + \tfrac{1}{2}SP = \tfrac{1}{2}r(PQ + QR + RS + SP).$$

Now using the property that the two tangents to a circle from a given point have equal length, it follows that $SP + QR = SZ + PZ + QX + RX = SY + PW + QW + RY = (PW + QW) + (RY + SY)$ $= PQ + RS = 25 + 25 = 50$ cm. Hence $PQ + QR + RS + SP = 50 + 50 = 100$ cm. Therefore, as the area of the trapezium of 600 cm^2, we have that $\frac{1}{2}r(100) = 600$ and hence $r = 12$.

21. Which of the following numbers does *not* have a square root in the form $x+y\sqrt{2}$, where x and y are positive integers?

A $17+12\sqrt{2}$ B $22+12\sqrt{2}$ C $38+12\sqrt{2}$ D $54+12\sqrt{2}$ E $73+12\sqrt{2}$

Solution: **D**

Suppose $a+12\sqrt{2}=(x+y\sqrt{2})^2$, where a, x and y are positive integers. It follows that $a+12\sqrt{2}=x^2+2xy\sqrt{2}+2y^2=(x^2+2y^2)+2xy\sqrt{2}$. Therefore $xy=6$ and $x^2+2y^2=a$. Since x and y are positive integers, the only possibilities for x and y are $x=1, y=6$; $x=2, y=3$; $x=3, y=2$ and $x=6, y=1$. Therefore the only possibilities for a are $1^2+2(6^2)=73$; $2^2+2(3^2)=22$; $3^2+2(2^2)=17$ and $6^2+2(1^2)=38$. Therefore options A, B, C and E are numbers which have square roots of the required form, but not option D.

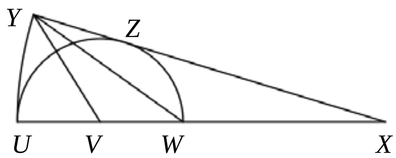
Extension Problems

21.1 Which numbers of the form $a+24\sqrt{2}$, where a is a positive integer, have square roots of the form $x+y\sqrt{2}$, where x and y are positive integers?

21.2 Which numbers of the form $a+24\sqrt{3}$, where a is a positive integer, have square roots of the form $x+y\sqrt{3}$, where x and y are positive integers.

21.2 Give an example of a number of the form $a+b\sqrt{5}$, where a and b are positive integers, which *does not* have a square root of the form $x+y\sqrt{5}$, where x and y are positive integers.

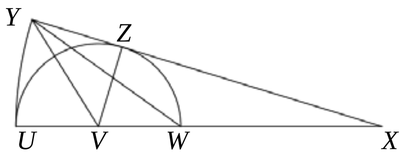
22. A semicircle of radius r is drawn with centre V and diameter UW. The line UW is then extended to the point X, such that the UW and WX are of equal length. An arc of the circle with centre X and radius $4r$ is then drawn so that the line XY is a tangent to the semicircle at Z, as shown. What, in terms of r, is the area of triangle YVW?



A $\frac{4r^2}{9}$ B $\frac{2r^2}{3}$ C r^2 D $\frac{4r^2}{3}$ E $2r^2$

Solution: **B**

Since $UW=WX$, we have $WX=UW=2r$. As V is the centre of the circle with UW as diameter, $VW=VZ=r$. Hence $VX=VW+WX=3r$. The triangles YVX and VVW have the same height and $VW=\frac{1}{3}VX$. Therefore $\text{area}(\Delta YVW)=\frac{1}{3}\text{area}(\Delta YVX)$. As VZ is a radius of the semicircle, and XY is a tangent to the semicircle at Z, the lines VZ and XY are perpendicular. Therefore $\text{area}(\Delta YVX)=\frac{1}{2}VZ.XY=\frac{1}{2}(r.4r)=2r^2$. Hence, $\text{area}(\Delta YVW)=\frac{1}{3}\text{area}(\Delta YVX)=\frac{1}{3}(2r^2)=\frac{2}{3}r^2$.

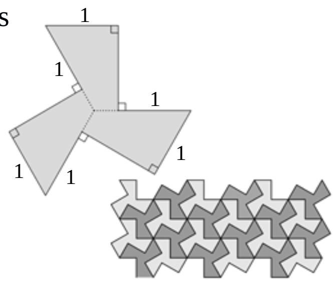


23.	Tom and Geri have a competition. Initially, each player has one attempt at hitting a target. If one player hits the target and the other does not then the successful player wins. If both players hit the target, or if both players miss the target, then each has another attempt, with the same rules applying. If the probability of Tom hitting the target is always $\frac{4}{5}$ and the probability of Geri hitting the target is always $\frac{2}{3}$, what is the probability that Tom wins the competition? A $\frac{4}{15}$ B $\frac{8}{15}$ C $\frac{2}{3}$ D $\frac{4}{5}$ E $\frac{13}{15}$

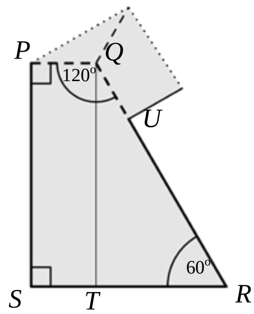
Solution: **C**

Let the probability that Tom wins the competition be *p*. The probability that initially Tom hits and Geri misses is $\frac{4}{5}\times\frac{1}{3}=\frac{4}{15}$. The probability that initially they both hit is $\frac{4}{5}\times\frac{2}{3}=\frac{8}{15}$ and that they both miss is $\frac{1}{5}\times\frac{1}{3}=\frac{1}{15}$. So the probability that either they both hit or both miss is $\frac{8}{15}+\frac{1}{15}=\frac{9}{15}=\frac{3}{5}$. If they both hit or both miss the competition is in the same position as it was initially. So Tom's probability of winning is then *p*. Therefore, $p=\frac{4}{15}+\frac{3}{5}p$. So $\frac{2}{5}p=\frac{4}{15}$ and hence $p=\frac{2}{3}$.

24.	The top diagram on the right shows a shape that tiles the plane, as shown in the lower diagram. The tile has nine sides, six of which have length 1. It may be divided into three congruent quadrilaterals as shown. What is the area of the tile? A $\frac{1+2\sqrt{3}}{2}$ B $\frac{4\sqrt{3}}{3}$ C $\sqrt{6}$ D $\frac{3+4\sqrt{3}}{4}$ E $\frac{3\sqrt{3}}{2}$



Solution: **B**



The tile is made up of three congruent quadrilaterals one of which, *PQRS*, is shown in the diagram on the right. We let *T* be the foot of the perpendicular from *Q* to *SR* and we let *U* be the point shown. Then $SR = UR = 1$.

Also *PQ*=*QU*. Let their common length be *x*. Hence $QR = 1 + x$ and $TR = 1 - x$. Because the three congruent quadrilaterals fit together at *Q*, we have $\angle PQR = 120^0$. As *PQ* and *SR* are both perpendicular to *PS*, they are parallel and hence $\angle QRT = 60^0$. Hence, from the right-angled triangle *QRT* we have $\frac{TR}{QR} = \sin 60^0$, that is, $\frac{1-x}{1+x}=\frac{1}{2}$, and therefore $2-2x = 1+x$. Hence $3x = 1$ and so $x=\frac{1}{3}$. Therefore $TR=\frac{2}{3}$ and $QR=\frac{4}{3}$. Now $QT/QR = \cos 60^0$ and hence $QT = QR\cos 60^0$ $=\frac{4}{3}\cos 60^0 = \frac{4}{3}(\frac{\sqrt{3}}{2}) = \frac{2\sqrt{3}}{3}$. [We could also use Pythagoras' Theorem applied to the triangle *QRT* to deduce this length.]

We can now work out the area of the quadrilateral *PQRS* in two different ways. Most straightforwardly, area(*PQRS*) = area(*PQTS*) + area(*QRT*) $=\frac{2\sqrt{3}}{3}\times\frac{1}{3}+\frac{1}{2}(\frac{2\sqrt{3}}{3}\times\frac{2}{3})=\frac{4\sqrt{3}}{9}$. Alternatively, as *PQ* is parallel to *SR*, *PQRS* is a trapezium. We now use the fact that the area of a trapezium is its height multiplied by the average length of its parallel sides. Hence, *PQRS* has area. $\frac{2\sqrt{3}}{3}\times\frac{1}{2}(\frac{1}{3}+1)$ $=\frac{2\sqrt{3}}{3}\times\frac{2}{3}=\frac{4\sqrt{3}}{9}$. The tile is made up of three copies of *PQRS*. So its area is $3\times\frac{4\sqrt{3}}{9}=\frac{4\sqrt{3}}{3}$.

25.	How many distinct pairs (x, y) of real numbers satisfy the equation $(x+y)^2 = (x+4)(y-4)$?				
	A 0	B 1	C 2	D 3	E 4

Solution: **B**

Method 1. The most straightforward method is to rewrite the given equation as a quadratic in *x*, whose coefficients involve *y*, and then use the "$b^2 - 4ac \geq 0$" condition for a quadratic, $ax^2 + bx + c = 0$, to have real number solutions.

Now, $(x+y)^2 = (x+4)(y-4) \Leftrightarrow x^2 + 2xy + y^2 = xy - 4x + 4y - 16$

$$\Leftrightarrow x^2 + xy + 4x + y^2 - 4y + 16 = 0 \ \Leftrightarrow x^2 + (y+4)x + (y^2 - 4y + 16) = 0.$$

So here $a = 1$, $b = y + 4$ and $c = y^2 - 4y + 16$. We see that $b^2 - 4ac = (y+4)^2 - 4(y^2 - 4y + 16)$ $= -3y^2 + 24y - 48 = -3(y^2 - 8y + 16) = -3(y-4)^2$. So there is a real number solution for *x* if and only if $-3(y-4)^2 \geq 0$. Now, as for all real numbers *y*, $(y-4)^2 \geq 0$, it follows that $-3(y-4)^2 \geq 0 \Leftrightarrow y - 4 = 0 \Leftrightarrow y = 4$. When $y = 4$ the quadratic equation becomes $x^2 + 8x + 16 = 0$, that is $(x+4)^2 = 0$, which has just the one solution $x = -4$. So $(-4, 4)$ is the only pair of real numbers, (x, y), for which $(x+y)^2 = (x+4)(y-4)$.

Method 2. We can simplify the algebra by making the substitution $w = x + 4$ and $z = y - 4$. Then $x + y = (w-4) + (z+4) = w + z$, and the equation becomes $(w+z)^2 = wz$.

Now, $(w+z)^2 = wz \Leftrightarrow w^2 + 2wz + z^2 = wz \Leftrightarrow w^2 + wz + z^2 = 0 \Leftrightarrow (w + \frac{1}{2}z)^2 + (\frac{\sqrt{3}}{2}z)^2 = 0$. The sum of the squares of two real numbers is zero if and only if each real number is 0. So the only real number solution of $(w + \frac{1}{2}z)^2 + (\frac{\sqrt{3}}{2}z)^2 = 0$ is $w + \frac{1}{2}z = \frac{\sqrt{3}}{2}z = 0$. This is equivalent to $w = z = 0$ and hence to $x = -4$ and $y = 4$. So again we deduce that there is just this one solution.

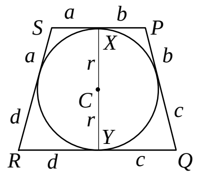
Extension Problems

25.1 It is possible to use the "$b^2 \geq 4ac$" criterion to show that $(0,0)$ is the only pair of real numbers that satisfy the equation $w^2 + wz + z^2 = 0$. Check this.

25.2 Show that the "$b^2 \geq 4ac$" criterion is correct by proving that for all real numbers, *a*, *b*, *c*, with $a \neq 0$, the quadratic equation $ax^2 + bx + c = 0$ has a real number solution if and only if $b^2 \geq 4ac$.

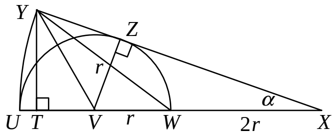
25.3 [For those who know about complex numbers.] If we allow the possibility that *x* and *y* are complex numbers, then the equation $(x+y)^2 = (x+4)(y-4)$ has more than one solution. Check that $x = 6$, $y = -1 + 5\sqrt{3}i$ is one solution of this equation. How many more can you find?

20. E The two tangents drawn from a point outside a circle to that circle are equal in length. This theorem has been used to mark four pairs of equal line segments on the diagram. In the circle the diameter, XY, has been marked. It is also a perpendicular height of the trapezium. We are given that $SR = PQ = 25$ cm so we can deduce that $(a + d) + (b + c) = 25 + 25 = 50$. The area of trapezium $PQRS = \frac{1}{2}(SP + QR) \times XY = 600 \text{ cm}^2$. Therefore $\frac{1}{2}(a + b + c + d) \times 2r = 600$. So $\frac{1}{2} \times 50 \times 2r = 600$, i.e. $r = 12$.



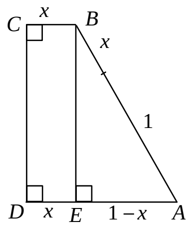
21. D $(x + y\sqrt{2})^2 = x^2 + 2xy\sqrt{2} + 2y^2$. Note that all of the alternatives given are of the form $a + 12\sqrt{2}$ so we need $xy = 6$. The only ordered pairs (x, y) of positive integers which satisfy this are (1, 6), (2, 3), (3, 2), (6, 1). For these, the values of $x^2 + 2y^2$ are 73, 22, 17, 38 respectively. So the required number is $54 + 12\sqrt{2}$.

22. B Let the perpendicular from Y meet UV at T and let $\angle ZXV = \alpha$. Note that $\angle VZX = 90°$ as a tangent to a circle is perpendicular to the radius at the point of contact. Therefore $\sin\alpha = \frac{r}{3r} = \frac{1}{3}$. Consider triangle YTX: $\sin\alpha = \frac{YT}{YX}$. So $YT = YX\sin\alpha = \frac{4r}{3}$. So the area of triangle $YVW = \frac{1}{2} \times VW \times YT = \frac{1}{2} \times r \times \frac{4r}{3} = \frac{2r^2}{3}$.



23. C Tom wins after one attempt each if he hits the target and Geri misses. The probability of this happening is $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$. Similarly the probability that Geri wins after one attempt is $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$. So the probability that both competitors will have at least one more attempt is $1 - \frac{4}{15} - \frac{2}{15} = \frac{3}{5}$.
Therefore the probability that Tom wins after two attempts each is $\frac{3}{5} \times \frac{4}{15}$. The probability that neither Tom nor Geri wins after two attempts each is $\frac{3}{5} \times \frac{3}{5}$. So the probability that Tom wins after three attempts each is $\left(\frac{3}{5}\right)^2 \times \frac{4}{15}$ and, more generally, the probability that he wins after n attempts each is $\left(\frac{3}{5}\right)^{n-1} \times \frac{4}{15}$.
Therefore the probability that Tom wins is $\frac{4}{15} + \left(\frac{3}{5}\right) \times \frac{4}{15} + \left(\frac{3}{5}\right)^2 \times \frac{4}{15} + \left(\frac{3}{5}\right)^3 \times \frac{4}{15} + \ldots$.
This is the sum to infinity of a geometric series with first term $\frac{4}{15}$ and common ratio $\frac{3}{5}$. Its value is $\frac{4}{15} \div \left(1 - \frac{3}{5}\right) = \frac{2}{3}$.

24. B The diagram shows one of the three quadrilaterals making up the tile, labelled and with a line BE inserted. Note that it is a trapezium. As three quadrilaterals fit together, it may be deduced that $\angle ABC = 360° \div 3 = 120°$, so $\angle BAD = 60°$. It may also be deduced that the length of AB is $1 + x$, where x is the length of BC. Now $\cos\angle BAD = \cos 60° = \frac{1}{2} = \frac{1-x}{1+x}$. So $1 + x = 2 - 2x$, i.e. $x = \frac{1}{3}$. The area of $ABCD$ is $\frac{1}{2}(AD + BC) \times CD = \frac{1}{2}\left(1 + \frac{1}{3}\right) \times \frac{4}{3}\sin 60°$ $= \frac{2}{3} \times \frac{4}{3} \times \frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{9}$. So the area of the tile is $3 \times \frac{4\sqrt{3}}{9} = \frac{4\sqrt{3}}{3}$.



25. B Starting with $(x + y)^2 = (x + 4)(y - 4)$ and expanding both sides gives $x^2 + 2xy + y^2 = xy - 4x + 4y - 16$, i.e. $x^2 + (y + 4)x + y^2 - 4y + 16 = 0$.
To eliminate the xy term we let $z = x + \frac{1}{2}y$ and then replace x by $z - \frac{1}{2}y$. The equation above becomes $z^2 + 4\left(z - \frac{1}{2}y\right) + \frac{3}{4}y^2 - 4y + 16 = 0$. However,

$$z^2 + 4\left(z - \tfrac{1}{2}y\right) + \tfrac{3}{4}y^2 - 4y + 16 = (z + 2)^2 + \tfrac{3}{4}y^2 - 6y + 12$$

$$= (z + 2)^2 + \tfrac{3}{4}\left(y^2 - 8y + 16\right) = (z + 2)^2 + \tfrac{3}{4}(y + 4)^2.$$

So the only real solution is when $z = -2$ and $y = 4$; i.e. $x = -4$ and $y = 4$.

1.	E
2.	B
3.	D
4.	B
5.	C
6.	C
7.	D
8.	C
9.	C
10.	E
11.	D
12.	E
13.	B
14.	D
15.	A
16.	A
17.	B
18.	A
19.	E
20.	E
21.	D
22.	B
23.	C
24.	B
25.	B

UKMT

UK SENIOR MATHEMATICAL CHALLENGE

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SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 6 NOVEMBER 2012

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **E** If an odd number is written as the sum of two prime numbers then one of those primes is 2, since 2 is the only even prime. However, 9 is not prime so 11 cannot be written as the sum of two primes. Note that 5 = 2 + 3; 7 = 2 + 5; 9 = 2 + 7; 10 = 3 + 7, so 11 is the only alternative which is not the sum of two primes.

2. **B** The interior angles of an equilateral triangle, square, regular pentagon are 60°, 90°, 108° respectively. The sum of the angles at a point is 360°. So $\theta = 360 - (60 + 90 + 108) = 102$.

3. **D** The cost now is $(70 + 4 \times 5 + 6 \times 2)$p = £1.02.

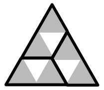
4. **B** One hundred thousand million is $10^2 \times 10^3 \times 10^6 = 10^{11}$. So the number of stars is $10^{11} \times 10^{11} = 10^{22}$.

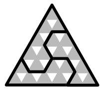
5. **C** Let the required addition be '*ab*' + '*cd*' + '*ef*', where a, b, c, d, e, f are single, distinct digits. To make this sum as large as possible, we need a, c, e (the tens digits) as large as possible; so they must be 7, 8, 9 in some order. Then we need b, d, f as large as possible, so 4, 5, 6 in some order. Hence the largest sum is $10(7+8+9)+(4+5+6) = 10 \times 24 + 15 = 255$.

6. **C** In order to be a multiple of 15, a number must be a multiple both of 3 and of 5. So its units digit must be 0 or 5. However, the units digit must also equal the thousands digit and this cannot be 0, so the required number is of the form '5*aa*5'. The largest such four-digit numbers are 5995, 5885, 5775. Their digit sums are 28, 26, 24 respectively. In order to be a multiple of 3, the digit sum of a number must also be a multiple of 3, so 5775 is the required number. The sum of its digits is 24.

7. **D** Add the first and third equations: $2x = 4$, so $x = 2$. Add the first two equations: $2x + 2y = 3$, so $y = -\frac{1}{2}$. Substitute for x and y in the first equation: $2 + \left(-\frac{1}{2}\right) + z = 1$ so $z = -\frac{1}{2}$. Therefore $xyz = 2 \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = \frac{1}{2}$.

8. **C** If an equilateral triangle is split into a number of smaller identical equilateral triangles then there must be one small triangle in the top row, three small triangles in the row below, five small triangles in the row below that and so on. So the total number of small triangles is 4 or 9 or 16 etc. These are all squares and it is left to the reader to prove that the sum of the first n odd numbers is n^2. So, for three copies of a given tile to form an equilateral triangle, the number of triangles which comprise the tile must be one third of a square number.



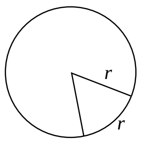


Only the tiles made up of three equilateral triangles and twelve equilateral triangles satisfy this condition. However, it is still necessary to show that three copies of these tiles can indeed make equilateral triangles. The diagrams above show how they can do this.

9. **C** If Pierre is telling the truth then Qadr is not telling the truth. However, this means that Ratna is telling the truth, so this leads to a contradiction as Pierre stated that just one person is telling the truth. So Pierre is not telling the truth, which means that Qadr is telling the truth, but Ratna is not telling the truth. This in turn means that Sven is telling the truth, but Tanya is not. So only Qadr and Sven are telling the truth.

10. **E** It can be deduced that N must consist of at least 224 digits since the largest 223-digit positive integer consists of 223 nines and has a digit sum of 2007. It is possible to find 224-digit positive integers which have a digit sum of 2012.
The largest of these is 99 999 ...999 995 and the smallest is 59 999 ...999 999.
So N = 59 999… 999 999 and $N + 1$ = 60 000… 000 000 (223 zeros).

11. **D** Let the radius of the circular piece of cardboard be r. The diagram shows a sector of the circle which would make one hat, with the minor arc shown becoming the circumference of the base of the hat. The circumference of the circle is $2\pi r$. Now $6r < 2\pi r < 7r$. This shows that we can cut out 6 hats in this fashion and also shows that the area of cardboard unused in cutting out *any* 6 hats is less than the area of a single hat. Hence there is no possibility that more than 6 hats could be cut out.



12. **E** Two different ways of expressing 5 are 1 + 4 and 4 + 1. In the following list these are denoted as {1, 4: two ways}. The list of all possible ways is {5: one way}, {2, 3: two ways}, {1, 4: two ways}, {1, 2, 2: three ways}, {1, 1, 3: three ways}, {1, 1, 1, 2: four ways}, {1, 1, 1, 1, 1: one way}. So in total there are 16 ways.
{*Different expressions of a positive integer in the above form are known as 'partitions'. It may be shown that the number of distinct compositions of a positive integer n is* 2^{n-1}.}

13. **B** The table below shows the position of the face marked with paint when the base of the cube is on the 25 squares. Code: T - top, B - base; F - front; H - hidden (rear); L - left; R - right.

1	2	**3**	4	5	6	**7**	8	9	10	**11**	12	13	14	**15**	16	17	18	19	**20**	21	22	23	**24**	25
T	H	**B**	F	T	R	**B**	L	T	F	**B**	H	T	L	**B**	R	R	R	R	**B**	L	L	L	**B**	F

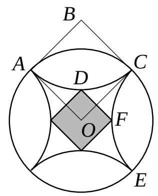
So the required sum is 3 + 7 + 11 + 15 + 20 + 24 = 80.

14. **D** Note that each student has a language in common with exactly four of the other five students. For instance, Jean-Pierre has a language in common with each of Ina, Karim, Lionel and Mary. Only Helga does not have a language in common with Jean-Pierre. So whichever two students are chosen, the probability that they have a language in common is 4/5.

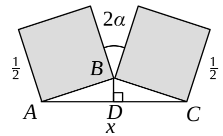
15. **A** Let Professor Rosseforp's usual journey take t minutes at an average speed of v metres/minute. Then the distance to work is vt metres. On Thursday his speed increased by 10%, i.e. it was $11v/10$ metres/minute. The time taken was $(t - x)$ minutes.
Therefore $11v/10 \times (t - x) = vt$. So $11(t - x) = 10t$, i.e. $t = 11x$.

16. **A** At points A and C, $x = 0$. So $y^2 - 4y = 12$, i.e. $(y-6)(y+2) = 0$, i.e. $y = 6$ or $y = -2$. So C is $(0, -2)$ and A is $(0, 6)$. At points B and D, $y = 0$. So $x^2 + x = 12$, i.e. $(x-3)(x+4) = 0$, i.e. $x = 3$ or $x = -4$. So D is $(-4, 0)$ and B is $(3, 0)$.
Therefore the areas of triangles DAB and DBC are $\frac{1}{2} \times 7 \times 6 = 21$ and $\frac{1}{2} \times 7 \times 2 = 7$.
So $ABCD$ has area 28. {*It is left to the reader to prove that area* $ABCD = \frac{1}{2}BD \times AC$.}

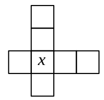
17. **B** In the diagram, B is the centre of the quarter-circle arc AC; D is the point where the central square touches arc AC; F is the point where the central square touches arc CE; O is the centre of the circle.
As both the circle and arc AC have radius 1, $OABC$ is a square of side 1.
By Pythagoras' Theorem: $OB^2 = 1^2 + 1^2$. So $OB = \sqrt{2}$. Therefore $OD = OB - DB = \sqrt{2} - 1$. By a similar argument, $OF = \sqrt{2} - 1$. Now $DF^2 = OD^2 + OF^2 = 2 \times OD^2$ since $OD = OF$. So the side of the square is $\sqrt{2} \times OD = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$.



18. **A** In the diagram, D is the midpoint of AC. Triangle ABC is isosceles since $AB = BC = \frac{1}{2}$. Therefore, BD bisects $\angle ABC$ and BD is perpendicular to AC. The angles at a point total 360°, so $\angle ABC = 360° - 2 \times 90° - 2\alpha = 180° - 2\alpha$. Therefore $\angle ABD = \angle CBD = 90° - \alpha$. So $\angle BAD = \angle BCD = \alpha$.
Therefore $x = AC = 2 \times AD = 2 \times AB \cos\alpha = 2 \times \frac{1}{2}\cos\alpha = \cos\alpha$.



19. **E** Note that the number represented by x appears in both the horizontal row and the vertical column. Note also that 2 + 3 + 4 + 5 + 6 + 7 + 8 = 35. Since the numbers in the row and those in the column have sum 21, we deduce that $x = 2 \times 21 - 35 = 7$.



We now need two disjoint sets of three numbers chosen from 2, 3, 4, 5, 6, 8 so that the numbers in both sets total 14. The only possibilities are {2, 4, 8} and {3, 5, 6}. We have six choices of which number to put in the top space in the vertical line, then two for the next space down and one for the bottom space. That leaves three choices for the first space in the horizontal line, two for the next space and one for the final space. So the total number of ways is $6 \times 2 \times 1 \times 3 \times 2 \times 1 = 72$.